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A Carlitz identity for the wreath product $C_r \wr \mathfrak{S}_n$

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A B S T R A C T

We present in this work a new flag major index fmaj_r for the wreath product $G_{r,n} = C_r \wr \mathfrak{S}_n$, where C_r is the cyclic group of order r and $\mathfrak{S}_n$ is the symmetric group on n letters. We prove that fmaj_r is equidistributed with the length function on $G_{r,n}$ and that the generating function of the pair $(\mathrm{des}_r, \mathrm{fmaj}_r)$ over $G_{r,n}$, where des_r is the usual descent number on $G_{r,n}$, satisfies a “natural” Carlitz identity, thus unifying and generalizing earlier results due to Carlitz (in the type A case), and Chow and Gessel (in the type B case). A q-Worpitzky identity, a convolution-type recurrence and a q-Frobenius formula are also presented, with combinatorial interpretation given to the expansion coefficients of the latter formula.

1. Introduction

Let $n \geqslant 1$ and let $\mathfrak{S}_n$ be the symmetric group on n letters. It is a classical result of Carlitz [7] that

$$\frac{\mathfrak{S}_n(t,q)}{\prod_{i=0}^{n}(1-tq^i)} = \sum_{k\geqslant 0}[k+1]_q^n t^k, \tag{1}$$

where $\mathfrak{S}_n(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}$, $\mathrm{des}(\sigma)$ and $\mathrm{maj}(\sigma)$ are respectively the descent number and major index of $\sigma \in \mathfrak{S}_n$, and $[n]_q$ is a q-integer. See Section 2 for definitions of undefined terms.

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When $q \to 1$, the Carlitz identity (1) becomes

$$\frac{\mathfrak{S}_n(t)}{(1-t)^{n+1}} = \sum_{k \geqslant 0} (k+1)^n t^k,$$

which is the rational generating function of the usual Eulerian polynomial $\mathfrak{S}_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$. Note here that $A_n(t) = t\mathfrak{S}_n(t)$, where $A_n(t)$ is the classical Eulerian polynomial enumerating $\mathfrak{S}_n$ by the number of increasing runs.

Generalizations of the Carlitz identity to other Coxeter families have also been made by various authors. See, e.g., [1,2,8] for the type B case, and [4] for the type D case. One can consider more generally the wreath product, which is denoted by $G_{r,n}$ in this work. Along this direction, Bagno and Biagioli [5] recently gave a generalized Carlitz identity for $G_{r,n}$. Since the classical Carlitz identity reduces to the rational generating function of $\mathfrak{S}_n(t)$, one may insist that a generalization of (1) ought to enjoy this property. In the type B case, only the generalized Carlitz identity due to Chow and Gessel [8] has this feature, whereas in other cases, the concerned generalized Carlitz identity all have the factor $[k+1]_q^n$ in place.

Steingrímsson had shown in [11] that

$$\frac{G_{r,n}(t)}{(1-t)^{n+1}} = \sum_{k \geqslant 0} (rk+1)^n t^k, \tag{2}$$

where $G_{r,n}$ and $G_{r,n}(t)$ are respectively denoted by S_n^r and $D(t)$. A "natural" Carlitz identity for $G_{r,n}$, if it exists, ought to have the factor $[rk+1]_q^n$ in place. Since $G_{1,n} = \mathfrak{S}_n$ and $G_{2,n} = B_n$, such a natural Carlitz identity will specialize to give the classical one for $\mathfrak{S}_n$ when $r = 1$, and the one for B_n when $r = 2$, where B_n is the nth hyperoctahedral group. It is the purpose of this work to show that such a natural Carlitz identity exists. More precisely, we show that the joint distribution of $(\mathrm{des}_r, \mathrm{fmaj}_r)$, where des_r is the usual descent number for $G_{r,n}$ and fmaj_r is a new notion of flag major index to be defined in Section 3, yields a natural q-version of (2). See Theorem 9 below.

The organization of this paper is as follows. In Section 2, we collect certain notations and results that we will need. In Section 3, we introduce fmaj_r, prove a combinatorial formula for it, and show that fmaj_r is equidistributed with the length function on $G_{r,n}$. In Section 4, we define a class of q-Eulerian polynomials $G_{r,n}(t,q)$ on $G_{r,n}$ and prove certain properties of them, including the generalized Carlitz identity. In Section 5, we prove further properties of $G_{r,n}(t,q)$, including a convolution-type recurrence and a q-Frobenius formula with combinatorial interpretion given to the expansion coefficients of the latter formula, and conjecture a separation property of the real zeros of $G_{r,n}(t,q)$ crucial for establishing the real-rootedness of $G_{r,n}(t,q)$. We conclude the present work in the final section by defining the notion of r-Euler–Mahonian of a pair of statistics on $G_{r,n}$ and mentioning one further line of research.

2. Notations and preliminaries

In this section we collect some definitions, notations, and results that will be used in subsequent sections of this paper. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers, $\mathbb{P} = \{1, 2, \ldots\}$ the set of all positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers, and $\mathbb{C}$ the set of all complex numbers.

The cardinality of a finite set S is denoted by $\#S$. For any $n \in \mathbb{Z}$, define $[n]$ to be the interval of integers $\{1, 2, \ldots, n\}$ if $n \geqslant 1$, and $\emptyset$ otherwise.

Here, we adopt those notions concerning the wreath product due to Steingrímsson [11]. Let $r, n \in \mathbb{P}$ and let $\mathfrak{S}_n$ denote the symmetric group on n letters. Any element π of $\mathfrak{S}_n$ is represented as the word $\pi(1)\pi(2)\cdots\pi(n)$. Let $C_r := \mathbb{Z}/r\mathbb{Z}$ be the cyclic group of order r, whose elements are represented by those of $\{0, 1, 2, \ldots, r-1\}$. Let $G_{r,n} := C_r \wr \mathfrak{S}_n$ be the wreath product of the symmetric group $\mathfrak{S}_n$ by the cyclic group C_r. Elements of $G_{r,n}$ are represented as $\pi \times \mathbf{z}$, where $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$

and $\mathbf{z} = (z_1, z_2, \dots, z_n)$ is an n-tuple of integers such that $z_i \in C_r$ for $i = 1, 2, \dots, n$. The product of elements $\pi \times \mathbf{z}$ and $\tau \times \mathbf{w}$ of $G_{r,n}$ is defined as: $(\pi \times \mathbf{z}) \cdot (\tau \times \mathbf{w}) = \pi\tau \times (\mathbf{w} + \tau(\mathbf{z}))$, where $\pi\tau = \pi \circ \tau$ is evaluated from right to left, $\tau(\mathbf{z}) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$ and the addition is coordinatewise modulo r.

It is easy to see that $e = 12\cdots n \times (0, 0, \dots, 0)$ is the identity element of $G_{r,n}$, where $12\cdots n$ is the identity element of $\mathfrak{S}_n$.

Definition 1. An integer $i \in [n]$ is called a *descent* of $p = \pi \times \mathbf{z} \in G_{r,n}$ if $z_i > z_{i+1}$ or $z_i = z_{i+1}$ and $\pi(i) > \pi(i+1)$, where $\pi(n+1) := n+1$ and $z_{n+1} := 0$. Denote by $D_r(p)$ the descent set of $p = \pi \times \mathbf{z} \in G_{r,n}$ and by $\operatorname{des}_r(p) := \#D_r(p)$ the number of descents (also called the *descent number*) of p.

For example, let $p = 436512 \times (3, 1, 0, 2, 2, 3) \in G_{4,6}$. Then $D_4(p) = \{1, 2, 4, 6\}$ so that $\operatorname{des}_4(p) = 4$.

Let q and t be two commuting indeterminates. We denote by $\mathbb{Q}[q]$ the ring of polynomials in q with rational coefficients and by $\mathbb{Q}[[q]]$ the corresponding ring of formal power series in q with rational coefficients. For $i \in \mathbb{P}$, the q-integer $[i]_q$ is defined as $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$; we also define $[0]_q := 0$. For $n \in \mathbb{N}$, the q-factorial $[n]_q!$ is defined as $[n]_q! := [1]_q[2]_q \cdots [n]_q$. For $0 \leqslant k \leqslant n$, the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We shall need a q-version of derivative. Define the Eulerian differential operator $\delta_{r,t} : \mathbb{Q}[[t,q]] \to \mathbb{Q}[[t,q]]$ by

$$\delta_{r,t} f(t,q) := \frac{f(tq^r, q) - f(t,q)}{t(q^r - 1)},$$

where $f(t,q) \in \mathbb{Q}[[t,q]]$. It is easy to see that $\delta_{r,t} t^n = [n]_{q^r} t^{n-1}$ and as $q \to 1$, $\delta_{r,t} t^n \to nt^{n-1}$, the usual derivative of t^n. A product rule for $\delta_{r,t}$ is as follows, whose proof is omitted.

Lemma 2. *We have*

$$\delta_{r,t}\big(A(t,q)B(t,q)\big) = \delta_{r,t}\big(A(t,q)\big)B\big(tq^r, q\big) + A(t,q)\delta_{r,t}\big(B(t,q)\big),$$

where $A(t,q), B(t,q) \in \mathbb{Q}[[t,q]]$.

A realization of the q-binomial theorem [3, Theorem 2.1] needed for the proof of Theorem 9 is the following identity

$$\sum_{k \geqslant 0} \begin{bmatrix} k+n \\ n \end{bmatrix}_q t^k = \frac{1}{(1-t)(1-tq)\cdots(1-tq^n)}, \tag{3}$$

which becomes the following binomial formula when $q = 1$:

$$\sum_{k \geqslant 0} \binom{k+n}{n} t^k = \frac{1}{(1-t)^{n+1}}.$$

3. Flag major index for $G_{r,n}$

The group $G_{r,n}$ is generated by $s_0, s_1, \dots, s_{n-1}$, where

$$s_0 = 12\cdots n \times (r-1, 0, \dots, 0),$$

and for $i = 1, 2, \dots, n-1$,

$$s_i = 1\cdots(i-1)(i+1)i(i+2)\cdots n \times (0, 0, \dots, 0).$$

These generators satisfy the following commutation relations:

$$\begin{gathered}(s_i s_{i+1})^3 = e \quad \text{where } 1 \leqslant i < n,\\ (s_i s_j)^2 = e \quad \text{where } |i-j| > 1,\\ s_0 s_i = s_i s_0 \quad \text{where } 1 < i < n,\\ (s_0 s_1)^{2r} = e.\end{gathered}$$

Using the number-theoretic fact that for $k = 1, 2, \dots, r$, $k(r-1) \equiv r-k \pmod r$, we have that $s_0^k = 12\cdots n \times (r-k, 0, \dots, 0)$ so that s_0 is of order r. When $r = 1$, s_0 is precisely the identity element e.

For $j = 1, 2, \dots, n$, let

$$\begin{aligned} t_j &= s_{j-1} s_{j-2} \cdots s_2 s_1 s_0 \\ &= j12\cdots(j-1)(j+1)\cdots n \times (r-1, 0, 0, \dots, 0). \end{aligned}$$

It is clear that $t_1, t_2, \dots, t_n$ form a set of generators in a distinguished flag of subgroups of $G_{r,n}$:

$$G_1 \subset G_2 \subset \cdots \subset G_n = G_{r,n},$$

where $G_i \cong G_{r,i}$ for $i = 1, 2, \dots, n$.

Each element $p = \pi \times \mathbf{z}$ of $G_{r,n}$ has a unique representation as a product

$$p = t_n^{k_n} t_{n-1}^{k_{n-1}} \cdots t_2^{k_2} t_1^{k_1}, \tag{4}$$

where $0 \leqslant k_j < rj$ for $j = 1, 2, \dots, n$.

Definition 3. For each $p = \pi \times \mathbf{z} \in G_{r,n}$, the *flag major index* of p is defined as

$$\mathrm{fmaj}_r(p) = k_1 + k_2 + \cdots + k_n.$$

The unique representation (4) of elements of $G_{r,n}$ insures that $\mathrm{fmaj}_r : G_{r,n} \to \mathbb{N}$ is well defined. It is known that the degrees of $G_{r,n}$ are $r, 2r, \dots, nr$ and that the Poincaré polynomial $P_{G_{r,n}}(q)$ of $G_{r,n}$ is given by

$$P_{G_{r,n}}(q) := \sum_{p \in G_{r,n}} q^{\ell(p)} = [r]_q [2r]_q \cdots [nr]_q,$$

where ℓ is the length function on $G_{r,n}$. See, e.g., Geck and Malle [9, Theorem 1.4, Table 1] and [10, §1.11]. A crucial property of fmaj_r is its equidistribution with ℓ on $G_{r,n}$, as follows.

Theorem 4. *We have*

$$\sum_{p\in G_{r,n}} q^{\mathrm{fmaj}_r(p)} = [r]_q[2r]_q\cdots[nr]_q.$$

Proof. By (4), every $p=\pi\times\mathbf{z}\in G_{r,n}$ can be written as $p=t_n^{k_n}\cdots t_2^{k_2}t_1^{k_1}$, where $k_1,\dots,k_n$ are uniquely determined integers satisfying $0\leqslant k_j<rj$ for $j=1,2,\dots,n$. Also, $\mathrm{fmaj}_r(p)=k_1+\cdots+k_n$. Thus,

$$\sum_{p\in G_{r,n}} q^{\mathrm{fmaj}_r(p)} = \prod_{j=1}^{n}\sum_{k_j=0}^{rj-1} q^{k_j} = [r]_q[2r]_q\cdots[nr]_q,$$

as desired. □

It makes perfect sense to define the major index of $p\in G_{r,n}$ as:

$$\mathrm{maj}(p) := \sum_{i=1}^{n} i\chi\big(i\in D_r(p)\big),$$

where $\chi(P)=1$ or 0 depending on whether the statement P is true or not. Note here that $D_r(p)$ includes n if it is a descent of p. In contrast to other previously considered cases, this makes a difference. The following combinatorial formula makes the computation of fmaj_r simple.

Theorem 5. *We have that for each $p=\pi\times\mathbf{z}\in G_{r,n}$,*

$$\mathrm{fmaj}_r(p) = r\,\mathrm{maj}(p) - \sum_{i=1}^{n} z_i. \tag{5}$$

Denote the right side of (5) by $\mathrm{maj}_{r,n}(p)$.

Lemma 6. *For any $p=\pi\times\mathbf{z}\in G_{r,n}$ with $\pi(n)\neq 1$ or $z_n\neq 1$, we have* $\mathrm{maj}_{r,n}(t_np)=\mathrm{maj}_{r,n}(p)+1$.

Proof. Let $p=\pi(1)\cdots\pi(n)\times(z_1,\dots,z_n)\in G_{r,n}$ and let $i_0\in[n]$ be the unique index such that $\pi(i_0)=1$. Then

$$\begin{aligned} t_np = {} & \big(\pi(1)-1\big)\cdots\big(\pi(i_0-1)-1\big)n\big(\pi(i_0+1)-1\big)\cdots\big(\pi(n)-1\big) \\ & \times(z_1,\dots,z_{i_0-1},r-1+z_{i_0},z_{i_0+1},\dots,z_n). \end{aligned}$$

Let $j\in[n]\setminus\{i_0-1,i_0\}$. We have $j\in D_r(t_np)\Leftrightarrow z_j>z_{j+1}$ or $z_j=z_{j+1}$ and $\pi(j)-1>\pi(j+1)-1\Leftrightarrow j\in D_r(p)$. For descents at positions i_0 and i_0-1, there are two cases to consider.

Case 1: $z_{i_0}=0$. Since $n>\pi(i_0+1)-1$ and $r-1+z_{i_0}=r-1\geqslant z_{i_0+1}$, $i_0\in D_r(t_np)$; since $\pi(i_0)=1$ and $z_{i_0}=0\not> z_{i_0+1}$, $i_0\notin D_r(p)$; since $\pi(i_0-1)-1<n$ and $z_{i_0-1}\leqslant r-1$, $i_0-1\notin D_r(t_np)$; since $\pi(i_0-1)>\pi(i_0)=1$ and $z_{i_0-1}\geqslant z_{i_0}=0$, $i_0-1\in D_r(p)$. Thus, we have $D_r(t_np)=\{i\in D_r(p)\colon\ i\neq i_0\}\cup\{i_0\}$ and $D_r(p)=\{i\in D_r(p)\colon\ i\neq i_0\}\cup\{i_0-1\}$ so that $\mathrm{maj}(t_np)=\mathrm{maj}(p)+1$ and hence

$$\begin{aligned}\operatorname{maj}_{r,n}(t_n p) &= r \operatorname{maj}(t_n p) - \sum_{1 \leqslant i \leqslant n,\ i \neq i_0} z_i - (r - 1 + z_{i_0}) \\ &= r\big(\operatorname{maj}(p) + 1\big) - \sum_{i=1}^{n} z_i - (r-1) \\ &= \operatorname{maj}_{r,n}(p) + 1.\end{aligned}$$

Case 2: $0 < z_{i_0} < r$. Since $n > \pi(i_0+1) - 1$ and $\pi(i_0) = 1 < \pi(i_0+1)$, $i_0 \in D_r(t_n p) \Leftrightarrow r - 1 + z_{i_0} \equiv z_{i_0} - 1 \geqslant z_{i_0+1} \Leftrightarrow z_{i_0} > z_{i_0+1} \Leftrightarrow i_0 \in D_r(p)$; since $\pi(i_0-1) - 1 < n$ and $\pi(i_0-1) > 1$, $i_0 - 1 \in D_r(t_n p) \Leftrightarrow z_{i_0-1} > z_{i_0} - 1 \Leftrightarrow z_{i_0-1} \geqslant z_{i_0} \Leftrightarrow i_0 - 1 \in D_r(p)$. Thus, we have $D_r(t_n p) = D_r(p)$ so that $\operatorname{maj}(t_n p) = \operatorname{maj}(p)$ and hence

$$\begin{aligned}\operatorname{maj}_{r,n}(t_n p) &= r \operatorname{maj}(t_n p) - \sum_{1 \leqslant i \leqslant n,\ i \neq i_0} z_i - (z_{i_0} - 1) \\ &= r \operatorname{maj}(p) - \sum_{i=1}^{n} z_i + 1 \\ &= \operatorname{maj}_{r,n}(p) + 1.\end{aligned}$$

In any case, we have $\operatorname{maj}_{r,n}(t_n p) = \operatorname{maj}_{r,n}(p) + 1$. $\square$

Proof of Theorem 5. Induction on n. Since $D_r(1 \times (z_1)) = \{1\}$ (resp., $\emptyset$) if $z_1 > 0$ (resp., $z_1 = 0$), by (5) we have that $\operatorname{fmaj}_r(1 \times (z_1)) = r - z_1$ (resp., 0). Also, $1 \times (z_1) = t_1^{r-z_1}$ (resp., t_1^0) if $z_1 > 0$ (resp., $z_1 = 0$) so that $\operatorname{maj}_{r,n}(1 \times (z_1)) = r - z_1$ (resp., 0) if $z_1 > 0$ (resp., $z_1 = 0$). Thus, the case $n = 1$ holds.

Assume now that the result holds for $n - 1$ (with $n \geqslant 2$) and let $p = \pi \times \mathbf{z} \in G_{r,n}$. There exist $0 \leqslant k_n < rn$ and $p' \in G_{r,n-1}$ such that $p = t_n^{k_n} p'$. By definition of fmaj_r, we have $\operatorname{fmaj}_r(p) = \operatorname{fmaj}_r(p') + k_n$. Since $p' = \tau \times \mathbf{w}$ can be identified as the element $\tilde{\tau} \times \tilde{\mathbf{w}}$ of $G_{r,n}$, where $\tilde{\tau}(i) = \tau(i)$ and $\tilde{w}_i = w_i$ for $i = 1, 2, \ldots, n-1$, $\tilde{\tau}(n) = n$ and $\tilde{w}_n = 0$, by definition of $\operatorname{maj}_{r,n}$, we have $\operatorname{maj}_{r,n}(p') = \operatorname{maj}_{r,n-1}(p')$. By induction, we have $\operatorname{fmaj}_r(p') = \operatorname{maj}_{r,n-1}(p')$ and hence

$$\operatorname{fmaj}_r(p) = \operatorname{fmaj}_r\big(p'\big) + k_n = \operatorname{maj}_{r,n}\big(p'\big) + k_n = \operatorname{maj}_{r,n}\big(t_n^{k_n} p'\big) = \operatorname{maj}_{r,n}(p),$$

where the next to last equality follows from iterations of Lemma 6. This finishes the induction and the proof. $\square$

4. Carlitz identity for $G_{r,n}$

We establish in this section the Carlitz identity for $G_{r,n}$. The following definition is crucial to the present work.

Definition 7. The q-Eulerian polynomial $G_{r,n}(t,q)$ of $G_{r,n}$ is defined by

$$G_{r,n}(t,q) = \sum_{p \in G_{r,n}} t^{\operatorname{des}_r(p)} q^{\operatorname{fmaj}_r(p)} = \sum_{k=0}^{n} G_{r,n,k}(q) t^k,$$

where $G_{r,n,k}(q) = \sum_{\operatorname{des}_r(p)=k} q^{\operatorname{fmaj}_r(p)}$.

The first four members of $G_{r,n}(t,q)$, computed according to Theorem 9(ii), are listed as follows.

$$G_{r,1}(t,q) = 1 + q[r-1]_q t,$$
$$G_{r,2}(t,q) = 1 + q\big([r-1]_q[r+1]_q + [2r-1]_q\big)t + q^{r+2}[r-1]_q^2 t^2,$$
$$\begin{aligned} G_{r,3}(t,q) = 1 &+ q\big([2r-1]_q[r+1]_q + [r+1]_q^2[r-1]_q + [3r-1]_q\big)t \\ &+ q^{r+2}\big([r-1]_q^2[2r+1]_q + [2r-1]_q^2 + [r-1]_q[r+1]_q[2r-1]_q\big)t^2 + q^{3r+3}[r-1]_q^3 t^3, \end{aligned}$$
$$\begin{aligned} G_{r,4}(t,q) = 1 &+ q\big([4r-1]_q + [r+1]_q^3[r-1] + [3r-1]_q[r+1]_q + [r+1]_q^2[2r-1]_q\big)t \\ &+ q^{r+2}\big([r+1]_q[2r-1]_q[3r-1]_q + [r-1]_q[r+1]_q[2r-1]_q[2r+1]_q \\ &+ [2r-1]_q^2[2r+1] + [r-1]_q[r+1]_q^2[3r-1]_q + [3r-1]_q^2 + [r-1]_q^2[2r+1]_q^2\big)t^2 \\ &+ q^{3r+3}\big([r-1]_q^2[2r-1]_q[2r+1]_q + [r-1]_q[r+1]_q[2r-1]_q^2 \\ &+ [r-1]_q^3[3r+1]_q + [2r-1]_q^3\big)t^3 + q^{6r+4}[r-1]_q^4 t^4. \end{aligned}$$

To facilitate the proof of Theorem 9, we gather in the next lemma several simple results, whose proofs are omitted.

Lemma 8. *The following hold*:

(i) $[rk+1]_q = 1 + q[r]_q[k]_{q^r}$;
(ii) $[rn-1]_q - [r]_q[k]_{q^r} = q^{rk}[rn-rk-1]_q$;
(iii) $[rk+1]_q[n]_{q^r} = [rj+1]_q[k+n-j]_{q^r} + q^{rj+1}[r(n-j)-1]_q[k-j]_{q^r}$.

The main theorem of this section is the following.

Theorem 9. *The following hold*:

(i) *for* $k = 1, 2, \ldots, n-1$,

$$G_{r,n,k}(q) = [rk+1]_q G_{r,n-1,k}(q) + q^{rk-(r-1)}\big[r(n-k) + (r-1)\big]_q G_{r,n-1,k-1}(q);$$

(ii) *for* $n \geqslant 1$,

$$G_{r,n}(t,q) = \big(qt[rn-1]_q + 1\big)G_{r,n-1}(t,q) + q[r]_q t(1-t)\delta_{r,t}\big(G_{r,n-1}(t,q)\big);$$

(iii) $$[rk+1]_q^n = \sum_{j=0}^{n} G_{r,n,j}(q)\begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r};$$

(iv) $$\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})} = \sum_{k\geqslant 0}[rk+1]_q^n t^k;$$

(v) $$\sum_{n\geqslant 0}\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}\frac{x^n}{n!} = \sum_{k\geqslant 0} t^k \exp\big([rk+1]_q x\big).$$

Proof. (i) Let $p = \pi \times \mathbf{z} \in G_{r,n-1}$, where $\pi = \pi(1)\cdots\pi(n-1) \in \mathfrak{S}_{n-1}$ and $\mathbf{z} = (z_1, \dots, z_{n-1}) \in C_r^{n-1}$. It is convenient to define $\pi(0) := 0$, $\pi(n) := n+1$, $z_0 := 0$ and $z_n := 0$. For $0 \leqslant i \leqslant n-1$, denote by

$$p_{i+1,j} = \pi(1)\cdots\pi(i)n\pi(i+1)\cdots\pi(n-1) \times (z_1, \dots, z_i, j, z_{i+1}, \dots, z_{n-1})$$

the element of $G_{r,n}$ obtained by inserting (n, j) to $p = \pi \times \mathbf{z} \in G_{r,n-1}$, where $j \in C_r$.

Since $\pi(i) < n > \pi(i+1)$ for $0 \leqslant i < n-1$, we have from the definition of descents that if $i \in D_r(p)$, then $z_i \geqslant z_{i+1}$ so that

$$\begin{aligned} i \in D_r(p_{i+1,j}) \quad \text{and} \quad i+1 \notin D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad z_i \geqslant z_{i+1} > j, \\ \text{both} \quad i, i+1 \in D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad z_i > j \geqslant z_{i+1}, \\ i+1 \in D_r(p_{i+1,j}) \quad \text{and} \quad i \notin D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad j \geqslant z_i \geqslant z_{i+1}; \end{aligned}$$

if $i \notin D_r(p)$, then $z_i \leqslant z_{i+1}$ so that

$$\begin{aligned} i \in D_r(p_{i+1,j}) \quad \text{and} \quad i+1 \notin D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad z_{i+1} \geqslant z_i > j, \\ \text{both} \quad i, i+1 \notin D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad z_{i+1} > j \geqslant z_i, \\ i+1 \in D_r(p_{i+1,j}) \quad \text{and} \quad i \notin D_r(p_{i+1,j}) \quad &\Leftrightarrow \quad j \geqslant z_{i+1} \geqslant z_i. \end{aligned}$$

Suppose that $D_r(p) = \{i_1, i_2, \dots, i_{k-1}\}_<$ and that the last descent of p is not equal to $n-1$. The set of non-descent positions is $\bigcup_{s=0}^{k-1}\{i_s+1, \dots, i_{s+1}-1\}$, where $i_0 := -1$ and $i_k := n$.

One may obtain elements of $G_{r,n}$ having k descents as follows.

For $s = 1, 2, \dots, k-1$ and $z_{i_s} > j \geqslant z_{i_s+1}$, we have

$$D_r(p_{i_s+1,j}) = \{i_1, \dots, i_s, i_s+1, i_{s+1}+1, \dots, i_{k-1}+1\}$$

so that $\mathrm{fmaj}_r(p_{i_s+1,j}) = \mathrm{fmaj}_r(p) + r(k-s+i_s) - j$. Summing over s and j, we get

$$\begin{aligned} q^{\mathrm{fmaj}_r(p)} \sum_{s=1}^{k-1} \sum_{j=z_{i_s+1}}^{z_{i_s}-1} q^{r(k-s+i_s)-j} = q^{\mathrm{fmaj}_r(p)} \Big\{ &\big(q^{r(k+i_1-1)-z_{i_1}+1} + \cdots + q^{r(k+i_1-1)-z_{i_1+1}}\big) \\ &+ \big(q^{r(k+i_2-2)-z_{i_2}+1} + \cdots + q^{r(k+i_2-2)-z_{i_2+1}}\big) + \cdots \\ &+ \big(q^{r(i_{k-1}+1)-z_{i_{k-1}}+1} + \cdots + q^{r(i_{k-1}+1)-z_{i_{k-1}+1}}\big) \Big\}. \qquad (6) \end{aligned}$$

For $s = 0, 1, \dots, k-1$, $l = i_s+1, \dots, i_{s+1}-1$ and $j \geqslant z_{l+1}$, we have

$$D_r(p_{l+1,j}) = \{i_1, \dots, i_s, l+1, i_{s+1}+1, \dots, i_{k-1}+1\}$$

so that $\mathrm{fmaj}_r(p_{l+1,j}) = \mathrm{fmaj}_r(p) + r(k-s+l) - j$. (When $l = i_k - 1 = n-1$, since $\pi(n-1) < n < \pi(n) = n+1$, n is a descent of $p_{n,j} \Leftrightarrow j > z_n = 0$. The sum over j below when $l = n-1$ is modified accordingly.) Summing over s, l and j, we get

$$
\begin{aligned}
&q^{\mathrm{fmaj}_r(p)}\sum_{s=0}^{k-1}\sum_{l=i_s+1}^{i_{s+1}-1}\sum_{j=z_{l+1}}^{r-1} q^{r(k-s+l)-j}\\
&\quad= q^{\mathrm{fmaj}_r(p)}\Big\{\Big[\big(q^{r(k-1)+1}+\cdots+q^{rk-z_1}\big)+\big(q^{rk+1}+\cdots+q^{r(k+1)-z_2}\big)\\
&\qquad+\cdots+\big(q^{r(k+i_1-3)+1}+\cdots+q^{r(k+i_1-2)-z_{i_1-1}}\big)+\big(q^{r(k+i_1-2)+1}+\cdots+q^{r(k+i_1-1)-z_{i_1}}\big)\Big]\\
&\qquad+\Big[\big(q^{r(k+i_1-1)+1}+\cdots+q^{r(k+i_1)-z_{i_1+2}}\big)+\big(q^{r(k+i_1)+1}+\cdots+q^{r(k+i_1+1)-z_{i_1+3}}\big)\\
&\qquad+\cdots+\big(q^{r(k+i_2-4)+1}+\cdots+q^{r(k+i_2-3)-z_{i_2-1}}\big)+\big(q^{r(k+i_2-3)+1}+\cdots+q^{r(k+i_2-2)-z_{i_2}}\big)\Big]+\cdots\\
&\qquad+\Big[\big(q^{r(i_{k-1}+1)+1}+\cdots+q^{r(i_{k-1}+2)-z_{i_{k-1}+2}}\big)+\big(q^{r(i_{k-1}+2)+1}+\cdots+q^{r(i_{k-1}+3)-z_{i_{k-1}+3}}\big)\\
&\qquad+\cdots+\big(q^{r(i_k-2)+1}+\cdots+q^{r(i_k-1)-z_{i_k-1}}\big)+\big(q^{r(i_k-1)+1}+\cdots+q^{ri_k-1}\big)\Big]\Big\}.
\end{aligned}
\tag{7}
$$

For $s=0,1,\ldots,k-1$, $l=i_s+1,\ldots,i_{s+1}-1$ and $j<z_l$, we have

$$D_r(p_{l+1,j})=\{i_1,\ldots,i_s,l,i_{s+1}+1,\ldots,i_{k-1}+1\}$$

so that $\mathrm{fmaj}_r(p_{l+1,j})=\mathrm{fmaj}_r(p)+r(k-1-s+l)-j$. (When $l=0$, there is not any j satisfying $0\leqslant j<z_l=0$. Thus, the middle sum below starts from $l=1$ when $s=0$.) Summing over s, l and j, we get

$$
\begin{aligned}
&q^{\mathrm{fmaj}_r(p)}\sum_{s=0}^{k-1}\sum_{l=i_s+1}^{i_{s+1}-1}\sum_{j=0}^{z_l-1} q^{r(k-1-s+l)-j}\\
&\quad= q^{\mathrm{fmaj}_r(p)}\Big\{\Big[\big(q^{rk-z_1+1}+\cdots+q^{rk}\big)+\big(q^{r(k+1)-z_2+1}+\cdots+q^{r(k+1)}\big)\\
&\qquad+\cdots+\big(q^{r(k+i_1-2)-z_{i_1-1}+1}+\cdots+q^{r(k+i_1-2)}\big)\Big]\\
&\qquad+\Big[\big(q^{r(k+i_1-1)-z_{i_1+1}+1}+\cdots+q^{r(k+i_1-1)}\big)+\big(q^{r(k+i_1)-z_{i_1+2}+1}+\cdots+q^{r(k+i_1)}\big)\\
&\qquad+\cdots+\big(q^{r(k+i_2-3)-z_{i_2-1}+1}+\cdots+q^{r(k+i_2-3)}\big)\Big]+\cdots\\
&\qquad+\Big[\big(q^{r(i_{k-1}+1)-z_{i_{k-1}+1}+1}+\cdots+q^{r(i_{k-1}+1)}\big)+\big(q^{r(i_{k-1}+2)-z_{i_{k-1}+2}+1}+\cdots+q^{r(i_{k-1}+2)}\big)\\
&\qquad+\cdots+\big(q^{r(i_k-1)-z_{i_k-1}+1}+\cdots+q^{r(i_k-1)}\big)\Big]\Big\}.
\end{aligned}
\tag{8}
$$

Summing (6), (7) and (8) and noting $ri_k-1=rn-1$, we obtain

$$q^{\mathrm{fmaj}_r(p)}\big(q^{rk-r+1}+q^{rk-r+2}+\cdots+q^{rn-1}\big)=q^{\mathrm{fmaj}_r(p)}q^{rk-(r-1)}\big[r(n-k)+(r-1)\big]_q. \tag{9}$$

By adapting the above arguments, one can show that (9) still holds when the last descent of p is equal to $n-1$. Summing now over all $p=\pi\times\mathbf{z}\in G_{r,n-1}$ having $k-1$ descents, the second term on the right side of the recurrence relation follows.

Suppose now $D_r(p)=\{i_1,\ldots,i_k\}_<$ and that the last descent of p is less than $n-1$. The set of non-descent positions is $\bigcup_{s=0}^{k}\{i_s+1,\ldots,i_{s+1}-1\}$, where $i_0:=-1$ and $i_{k+1}:=n$.

One may obtain elements of $G_{r,n}$ having k descents as follows.

For $s=1,2,\ldots,k$ and $z_{i_s+1}>j$, we have

$$D_r(p_{i_s+1,j})=\{i_1,\ldots,i_s,i_{s+1}+1,\ldots,i_k+1\}$$

so that $\mathrm{fmaj}_r(p_{i_s+1,j}) = \mathrm{fmaj}_r(p) + r(k-s) - j$. Since $i_k+1,\dots,i_{k+1}-1 = n-1$ are non-descent positions, $z_{i_k+1} \leqslant \cdots \leqslant z_{n-1} = 0$ so that there is not any $0 \leqslant j < r-1$ satisfying $0 \leqslant j < z_{i_k+1} = 0$. Summing over s and j, we get

$$\begin{aligned} &q^{\mathrm{fmaj}_r(p)} \sum_{s=1}^{k-1} \sum_{j=0}^{z_{i_s+1}-1} q^{r(k-s)-j} \\ &\quad = q^{\mathrm{fmaj}_r(p)} \bigl\{ \bigl(q^{r(k-1)-z_{i_1+1}+1} + \cdots + q^{r(k-1)}\bigr) + \bigl(q^{r(k-2)-z_{i_2+1}+1} + \cdots + q^{r(k-2)}\bigr) + \cdots \\ &\qquad + \bigl(q^{r-z_{i_{k-1}+1}+1} + \cdots + q^r\bigr) \bigr\}. \end{aligned} \tag{10}$$

For $s = 1,2,\dots,k$ and $j \geqslant z_{i_s}$, we have

$$D_r(p_{i_s+1,j}) = \{i_1,\dots,i_{s-1}, i_s+1,\dots,i_k+1\}$$

so that $\mathrm{fmaj}_r(p_{i_s+1,j}) = \mathrm{fmaj}_r(p) + r(k-s+1) - j$. Summing over s and j, we get

$$\begin{aligned} &q^{\mathrm{fmaj}_r(p)} \sum_{s=1}^{k} \sum_{j=z_{i_s}}^{r-1} q^{r(k-s+1)-j} \\ &\quad = q^{\mathrm{fmaj}_r(p)} \bigl\{ \bigl(q^{r(k-1)+1} + \cdots + q^{rk-z_{i_1}}\bigr) + \bigl(q^{r(k-2)+1} + \cdots + q^{r(k-1)-z_{i_2}}\bigr) + \cdots \\ &\qquad + \bigl(q + \cdots + q^{r-z_{i_k}}\bigr) \bigr\}. \end{aligned} \tag{11}$$

For $s = 0,1,\dots,k$, $l = i_s+1,\dots,i_{s+1}-1$ and $z_{l+1} > j \geqslant z_l$, we have

$$D_r(p_{l+1,j}) = \{i_1,\dots,i_s, i_{s+1}+1,\dots,i_k+1\}$$

so that $\mathrm{fmaj}_r(p_{l+1,j}) = \mathrm{fmaj}_r(p) + r(k-s) - j$.

Since $i_k+1,\dots,i_{k+1}-1 = n-1$ are non-descents, $z_{i_k+1} \leqslant \cdots \leqslant z_{i_{k+1}-1} = z_{n-1} = 0$ so that there is not any $0 \leqslant j < r$ satisfying $z_{l+1} > j \geqslant z_l$ for $l = i_k+1,\dots,i_{k+1}-2 = n-2$. On the other hand, since $\pi(n-1) < n < \pi(n) = n+1$ and $n-1 \notin D_r(p)$, $0 = z_{n-1} \leqslant j \geqslant z_n = 0$ so that $n-1 \notin D_r(p_{n,j})$ for any $0 \leqslant j < r-1$. Moreover, $n \notin D_r(p_{n,j}) \Leftrightarrow j = 0$.

Thus, for $s = k$, only $l = i_{k+1}-1 = n-1$ and $j = 0$ gives rise to an element $p_{n,0}$ of $G_{r,n}$ having $D_r(p_{n,0}) = D_r(p)$ and $\mathrm{fmaj}_r(p_{n,0}) = \mathrm{fmaj}_r(p)$.

Summing over s, l and j, we get

$$\begin{aligned} &q^{\mathrm{fmaj}_r(p)} \sum_{s=0}^{k} \sum_{l=i_s+1}^{i_{s+1}-1} \sum_{j=z_l}^{z_{l+1}-1} q^{r(k-s)-j} \\ &\quad = q^{\mathrm{fmaj}_r(p)} \bigl\{ \bigl(q^{rk-z_{i_1}+1} + \cdots + q^{rk}\bigr) + \bigl(q^{r(k-1)-z_{i_2}+1} + \cdots + q^{r(k-1)-z_{i_1+1}}\bigr) \\ &\qquad + \cdots + \bigl(q^{r-z_{i_k}+1} + \cdots + q^{r-z_{i_{k-1}+1}}\bigr) + 1 \bigr\}, \end{aligned} \tag{12}$$

where successive terms on the right correspond to $s = 0,1,\dots,k$. Summing (10), (11) and (12), we get

$$q^{\mathrm{fmaj}_r(p)}\bigl(1 + q + q^2 + \cdots + q^{rk}\bigr) = q^{\mathrm{fmaj}_r(p)}[rk+1]_q. \tag{13}$$

By adapting the above arguments, one can show that (13) still holds when the last descent of p is equal to $n-1$. Summing now over all $p = \pi \times \mathbf{z} \in G_{r,n-1}$ having k descents, we obtain the first term on the right side of the recurrence relation. Combining both terms, (i) follows.

(ii) Multiplying the recurrence relation in (i) by t^k followed by summing over $k \geqslant 1$, and using Lemma 8, we have

$$\begin{aligned}
\sum_{k\geqslant 1} G_{r,n,k}(q)t^k &= \sum_{k\geqslant 1}\bigl\{[rk+1]_q G_{r,n-1,k}(q) + q^{rk-(r-1)}\bigl[r(n-k)+(r-1)\bigr]_q G_{r,n-1,k-1}(q)\bigr\}t^k \\
&= \sum_{k\geqslant 1}[rk+1]_q G_{r,n-1,k}(q)t^k + \sum_{k\geqslant 0} q^{rk+1}[rn-rk-1]_q G_{r,n-1,k}(q)t^{k+1} \\
&= \sum_{k\geqslant 1}\bigl(1+q[r]_q[k]_{q^r}\bigr)G_{r,n-1,k}(q)t^k + \sum_{k\geqslant 0} q\bigl([rn-1]_q - [r]_q[k]_{q^r}\bigr)G_{r,n-1,k}(q)t^{k+1} \\
&= \sum_{k\geqslant 0}\bigl(qt[rn-1]_q+1\bigr)G_{r,n-1,k}(q)t^k - 1 + q[r]_q t(1-t)\sum_{k\geqslant 0}[k]_{q^r} G_{r,n-1,k}(q)t^{k-1} \\
&= \bigl(qt[rn-1]_q+1\bigr)G_{r,n-1}(t,q) - 1 + q[r]_q t(1-t)\delta_{r,t}\bigl(G_{r,n-1}(t,q)\bigr).
\end{aligned}$$

Since $G_{r,n,0}(q)=1$, transposing the 1 to the left side, (ii) follows.

(iii) Induction on n. Since $G_{r,1}(t,q) = 1 + q[r-1]_q t$, we clearly have

$$\begin{aligned}
\begin{bmatrix} k+1 \\ 1 \end{bmatrix}_{q^r} G_{r,1,0}(q) + \begin{bmatrix} k \\ 1 \end{bmatrix}_{q^r} G_{r,1,1}(q) &= [k+1]_{q^r} + q[r-1]_q[k]_{q^r} \\
&= [rk+1]_q
\end{aligned}$$

so that the case $n=1$ holds. Suppose now that

$$[rk+1]_q^{n-1} = \sum_{j=0}^{n-1} G_{r,n-1,j}(q)\begin{bmatrix} k+n-1-j \\ n-1 \end{bmatrix}_{q^r}$$

holds, where $n \geqslant 2$. Multiplying both sides by $[rk+1]_q$, we have

$$\begin{aligned}
[rk+1]_q^n &= \sum_{j=0}^{n-1} G_{r,n-1,j}(q)\begin{bmatrix} k+n-1-j \\ n-1 \end{bmatrix}_{q^r}[rk+1]_q \\
&= \sum_{j=0}^{n-1} G_{r,n-1,j}(q)\begin{bmatrix} k+n-1-j \\ n-1 \end{bmatrix}_{q^r} \\
&\quad \times \frac{[rj+1]_q[k+n-j]_{q^r} + q^{rj+1}[r(n-j)-1]_q[k-j]_{q^r}}{[n]_{q^r}} \\
&= \sum_{j=0}^{n-1}[rj+1]_q G_{r,n-1,j}(q)\begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r} \\
&\quad + \sum_{j=0}^{n-1} q^{rj+1}\bigl[r(n-j)-1\bigr]_q G_{r,n-1,j}(q)\begin{bmatrix} k+n-1-j \\ n \end{bmatrix}_{q^r}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{n-1} [rj+1]_q G_{r,n-1,j}(q) \begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r} \\
&\quad + \sum_{j=1}^{n} q^{rj-(r-1)} \big[r(n-j)+(r-1)\big]_q G_{r,n-1,j-1}(q) \begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r} \\
&= \sum_{j=0}^{n} G_{r,n,j}(q) \begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r},
\end{aligned}$$

where the second equality follows from Lemma 8(iii). This finishes the induction and the proof of (iii).

(iv) Multiplying (iii) by t^k followed by summing over k, we have

$$\begin{aligned}
\sum_{k\geqslant 0} [rk+1]_q^n t^k &= \sum_{k\geqslant 0} \sum_{j=0}^{n} G_{r,n,j}(q) \begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r} t^k \\
&= \sum_{j=0}^{n} G_{r,n,j}(q) t^j \sum_{k\geqslant j} \begin{bmatrix} k+n-j \\ n \end{bmatrix}_{q^r} t^{k-j} \\
&= \frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})},
\end{aligned}$$

where the last equality follows from (3).

(v) Multiplying (iv) by $x^n/n!$ followed by summing over $n \geqslant 0$, we have

$$\sum_{n\geqslant 0} \frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})} \frac{x^n}{n!} = \sum_{k\geqslant 0} t^k \sum_{n\geqslant 0} \frac{([rk+1]_q x)^n}{n!} = \sum_{k\geqslant 0} t^k \exp\big([rk+1]_q x\big). \qquad \square$$

Theorem 9(iii) is a *q-Worpitzky identity*. When $q = 1$, Theorem 9(i) and (iv) specialize to give [11, Lemma 16, Theorem 17], and Theorem 9(v), upon replacing x by $x(1-t)$ followed by multiplication by $(1-t)$, becomes

$$\sum_{n\geqslant 0} G_{r,n}(t) \frac{x^n}{n!} = (1-t) \sum_{k\geqslant 0} t^k e^{(rk+1)x(1-t)} = \frac{(1-t)e^{x(1-t)}}{1-te^{rx(1-t)}},$$

which is [11, Theorem 20].

5. Further properties of $G_{r,n}(t,q)$

We present in this section further properties of $G_{r,n}(t,q)$. A convolution-type recurrence relation for $G_{r,n}(t,q)$ is as follows.

Proposition 10. *We have for $n \geqslant 1$,*

$$G_{r,n}(t,q) = \sum_{i=1}^{n} \binom{n-1}{i-1} q^{ri-(r-1)} [r]_q^{n-i+1} t G_{r,i-1}(t,q) G_{1,n-i}\big(tq^{ri}, q^r\big) + \big(1-tq^{rn}\big) G_{r,n-1}(t,q).$$

Proof. By Lemma 8 and Theorem 9(iv), we have

$$\begin{aligned}
&\frac{\sum_{i=1}^n \binom{n-1}{i-1} q^{ri-(r-1)}[r]_q^{n-i+1} t G_{r,i-1}(t,q) G_{1,n-i}(tq^{ri},q^r)}{\prod_{j=0}^n (1-tq^{rj})} \\
&= \sum_{i=1}^n \binom{n-1}{i-1} q^{ri-(r-1)}[r]_q^{n-i+1} t \left(\frac{G_{r,i-1}(t,q)}{\prod_{j=0}^{i-1}(1-tq^{rj})}\right)\left(\frac{G_{1,n-i}(tq^{ri},q^r)}{\prod_{j=i}^{n}(1-tq^{rj})}\right) \\
&= \sum_{i=1}^n \binom{n-1}{i-1} q^{ri-(r-1)}[r]_q^{n-i+1} t \left(\sum_{k\geqslant 0}[rk+1]_q^{i-1}t^k\right)\left(\sum_{l\geqslant 0}[l+1]_{q^r}^{n-i}\big(tq^{ri}\big)^l\right) \\
&= \sum_{m\geqslant 0} t^{m+1} \sum_{\substack{k+l=m\\ 0\leqslant k,l\leqslant m}} q^{rl+1}[r]_q \sum_{i=1}^n \binom{n-1}{i-1}\big(q^{r(l+1)}[rk+1]_q\big)^{i-1}\big([r]_q[l+1]_{q^r}\big)^{n-i} \\
&= \sum_{m\geqslant 0} t^{m+1} \sum_{\substack{k+l=m\\ 0\leqslant k,l\leqslant m}} q^{rl+1}[r]_q\big(q^{r(l+1)}[rk+1]_q + [r]_q[l+1]_{q^r}\big)^{n-1} \\
&= \sum_{m\geqslant 0} t^{m+1} q[r]_q \sum_{l=0}^m q^{rl}[rm+r+1]_q^{n-1} \\
&= \sum_{m\geqslant 0} t^{m+1} q[r]_q[m+1]_{q^r}[rm+r+1]_q^{n-1} \\
&= \sum_{m\geqslant 0}[rm+1]_q^n t^m - \sum_{m\geqslant 0}[rm+1]_q^{n-1}t^m \\
&= \frac{G_{r,n}(t,q)-(1-tq^{rn})G_{r,n-1}(t,q)}{\prod_{i=0}^n(1-tq^{ri})}.
\end{aligned}$$

This finishes the proof. □

A combinatorial proof of the preceding proposition is also possible. See, e.g., [6, Theorem 3.6], [8, Theorem 4.4] for samples of arguments in the type B case.

The following is a q-Frobenius formula for $G_{r,n}(t,q)$.

Proposition 11. *We have*

$$\frac{G_{r,n}(t,q)}{\prod_{i=0}^n(1-tq^{ri})} = \sum_{k=0}^n \frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^k(1-tq^{ri})},$$

where the polynomial $S_{n,k}^r(q)$ in q satisfies the recurrence relation

$$S_{n+1,k}^r(q) = [rk+1]_q S_{n,k}^r(q) + q^{rk-(r-1)}[r]_q S_{n,k-1}^r(q). \tag{14}$$

Proof. For $k = 0, 1, \ldots, n$, the polynomials

$$\theta_k(t) = t^k\big(1-tq^{r(k+1)}\big)\big(1-tq^{r(k+2)}\big)\cdots\big(1-tq^{rn}\big)$$

in t are linearly independent and of degree n so that $G_{r,n}(t,q)=\sum_{k=0}^{n} C_{n,k}(q)\theta_k(t)$ for some polynomial $C_{n,k}(q)$ in q. Now define $S_{n,k}^r(q)=C_{n,k}^r(q)/[k]_{q^r}!$ so that

$$\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}=\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}. \tag{15}$$

Applying the Eulerian differential operator $\delta_{r,t}$ to (15), we have

$$\delta_{r,t}\left(\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}\right)=\frac{[rn+r]_q G_{r,n}(t,q)+[r]_q(1-t)\delta_{r,t}(G_{r,n}(t,q))}{[r]_q\prod_{i=0}^{n+1}(1-tq^{ri})}$$

so that

$$q[r]_q t\delta_{r,t}\left(\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}\right)=\frac{qt[rn+r]_q G_{r,n}(t,q)+q[r]_q t(1-t)\delta_{r,t}(G_{r,n}(t,q))}{\prod_{i=0}^{n+1}(1-tq^{ri})}.$$

Since $1+qt[rn+r-1]_q=(1-tq^{rn+r})+qt[rn+r]_q$, by Theorem 9(ii), we have

$$\frac{G_{r,n+1}(t,q)}{\prod_{i=0}^{n+1}(1-tq^{ri})}=\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}+q[r]_q t\delta_{r,t}\left(\frac{G_{r,n}(t,q)}{\prod_{i=0}^{n}(1-tq^{ri})}\right). \tag{16}$$

Since $[rk]_q=[r]_q[k]_{q^r}$, replacing both sides of (16) by (15), we have

$$\begin{aligned}
&\sum_{k=0}^{n+1}\frac{[k]_{q^r}!S_{n+1,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}\\
&\quad=\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}+q[r]_q t\delta_{r,t}\left(\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}\right)\\
&\quad=\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}+q[r]_q t\left(\sum_{k=0}^{n}\frac{[k]_{q^r}![rk+r]_q q^{rk}S_{n,k}^r(q)t^k}{[r]_q\prod_{i=0}^{k+1}(1-tq^{ri})}+\sum_{k=0}^{n}\frac{[k]_{q^r}![k]_{q^r}S_{n,k}^r(q)t^{k-1}}{\prod_{i=0}^{k}(1-tq^{ri})}\right)\\
&\quad=\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}+\sum_{k=0}^{n}\frac{[k]_{q^r}![rk+r]_q q^{rk+1}S_{n,k}^r(q)t^{k+1}}{\prod_{i=0}^{k+1}(1-tq^{ri})}+\sum_{k=0}^{n}\frac{q[r]_q[k]_{q^r}![k]_{q^r}S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}\\
&\quad=\sum_{k=0}^{n}\frac{[k]_{q^r}!S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}+\sum_{k=0}^{n+1}\frac{[k-1]_{q^r}![rk]_q q^{rk-(r-1)}S_{n,k-1}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}+\sum_{k=0}^{n}\frac{q[r]_q[k]_{q^r}![k]_{q^r}S_{n,k}^r(q)t^k}{\prod_{i=0}^{k}(1-tq^{ri})}\\
&\quad=\sum_{k=0}^{n+1}\frac{[k]_{q^r}!\{(1+q[r]_q[k]_{q^r})S_{n,k}^r(q)+q^{rk-(r-1)}[r]_q S_{n,k-1}^r(q)\}t^k}{\prod_{i=0}^{k}(1-tq^{ri})}\\
&\quad=\sum_{k=0}^{n+1}\frac{[k]_{q^r}!\{[rk+1]_q S_{n,k}^r(q)+q^{rk-(r-1)}[r]_q S_{n,k-1}^r(q)\}t^k}{\prod_{i=0}^{k}(1-tq^{ri})},
\end{aligned}$$

where the second equality follows from an application of Lemma 2. Equating the coefficients of

$t^k/(1-t)(1-tq^r)\cdots(1-tq^{rk})$, we get

$$S^r_{n+1,k}(q)=[rk+1]_q S^r_{n,k}(q)+q^{rk-(r-1)}[r]_q S^r_{n,k-1}(q).$$

Since $S^r_{n,0}(q)=1$ for $n\geqslant 0$, the above recurrence implies that $S^r_{n,k}(q)$ is a polynomial in q. This finishes the proof. $\square$

The values of $S^r_{n,k}(q)$ for $n=1,2,3,4$ are as follows

$$\begin{gathered}
S^r_{1,1}(q)=q[r]_q,\\
S^r_{2,1}(q)=q[r]_q\big([r+1]_q+1\big),\\
S^r_{2,2}(q)=q^{r+2}[r]_q^2,\\
S^r_{3,1}(q)=q[r]_q\big([r+1]_q^2+[r+1]_q+1\big),\\
S^r_{3,2}(q)=q^{r+2}[r]_q^2\big([2r+1]_q+[r+1]_q+1\big),\\
S^r_{3,3}(q)=q^{3r+3}[r]_q^3,\\
S^r_{4,1}(q)=q[r]_q\big([r+1]_q^3+[r+1]_q^2+[r+1]_q+1\big),\\
S^r_{4,2}(q)=q^{r+2}[r]_q^2\big([2r+1]_q^2+[2r+1]_q[r+1]_q+[r+1]_q^2+[2r+1]_q+[r+1]_q+1\big),\\
S^r_{4,3}(q)=q^{3r+3}[r]_q^3\big([3r+1]_q+[2r+1]_q+[r+1]_q+1\big),\\
S^r_{4,4}(q)=q^{6r+4}[r]_q^4,
\end{gathered}$$

and $S^r_{n,0}(q)=1$ for all n.

The polynomials $S^r_{n,k}(q)$ have a combinatorial characterization. Let S be a set of positive integers and ζ be a primitive rth root of unity. An r-signed partition of S is a collection $\pi=(B_1,\dots,B_k)$ of subsets of $\bigcup_{j=0}^{r-1}\zeta^j S$ with $\min|B_1|\leqslant\cdots\leqslant\min|B_k|$ and such that $\{\zeta^j B_i\colon\ i\in[k],\ 0\leqslant j<r\}$ is a partition of $\bigcup_{j=0}^{r-1}\zeta^j S$, where $\zeta^j S=\{\zeta^j s\colon\ s\in S\}$ and $|S|=\{|s|\colon\ s\in S\}$.

We call $B_1,\dots,B_k$ the *blocks* of π and say that π has k blocks. We also let

$$C(\pi)=\sum_{x\in\bigcup_{i=1}^k B_i}(r-j)\chi\big(x\in\zeta^j S\big).$$

A *partial r-signed partition* of S is an r-signed partition of some subset of S. Denote by $\Pi_{r,\subseteq}(S,k)$ the set of all partial r-signed partitions of S having k blocks, and let

$$S_r\big([n],k,q\big)=\sum_{\pi\in\Pi_{r,\subseteq}([n],k)}q^{m(\pi)},$$

where

$$m(\pi)=r\sum_{i=1}^k(i-1)\sum_{\nu=1}^n\chi\left(\nu\in\bigcup_{j=0}^{r-1}\zeta^j B_i\right)+C(\pi).$$

Proposition 12. *We have* $S^r_{n,k}(q)=S_r([n],k,q)$.

Proof. It suffices to show that $S_r([n],k,q)$ satisfies (14) and the initial condition. The case $n=1$ is trivial. So, we let $n>1$ and $\pi=(B_1,\ldots,B_k)\in\Pi_{r,\subseteq}([n],k)$. If $\{\zeta^j n\}$ is a block of π, then $\{\zeta^j n\}=B_k$ and removing it from π yields a partial r-signed partition τ of $[n-1]$ into $k-1$ blocks with $C(\pi)=C(\tau)+r-j$ and $m(\pi)=m(\tau)+r(k-1)+(r-j)$.

If $\{\zeta^j n\}\subsetneqq B_i$ for some $i\in[k]$, then removing $\zeta^j n$ from B_i yields a partial r-signed partition τ' of $[n-1]$ into k blocks with $C(\pi)=C(\tau')+r-j$ and $m(\pi)=m(\tau')+r(i-1)+(r-j)$.

If none of $\zeta^0 n,\zeta^1 n,\ldots,\zeta^{r-1}n$ lies in any block of π, then $\pi\in\Pi_{r,\subseteq}([n-1],k)$. Thus,

$$\begin{aligned}
S_r\big([n],k,q\big) &= \sum_{j=0}^{r-1}\sum_{\tau\in\Pi_{r,\subseteq}([n-1],k-1)} q^{m(\tau)+r(k-1)+(r-j)} \\
&\quad + \sum_{i=1}^{k}\sum_{j=0}^{r-1}\sum_{\tau'\in\Pi_{r,\subseteq}([n-1],k)} q^{m(\tau')+r(i-1)+(r-j)} + \sum_{\pi\in\Pi_{r,\subseteq}([n-1],k)} q^{m(\pi)} \\
&= q^{rk-(r-1)}[r]_q S_r\big([n-1],k-1,q\big) + \big(q[r]_q[k]_{q^r}+1\big)S_r\big([n-1],k,q\big) \\
&= [rk+1]_q S_r\big([n-1],k,q\big) + q^{rk-(r-1)}[r]_q S_r\big([n-1],k-1,q\big).
\end{aligned}$$

This finishes the proof. $\square$

We now address the real-rootedness of $G_{r,n}(t,q)$. The following conjecture, which generalizes [8, Conjectures 4.6, 4.8] for types A and B cases, and has been verified computationally for $1\leqslant r,n\leqslant 20$ and for $q=0.2,0.4,\ldots,10$, if true, would imply $G_{r,n}(t,q)$ interlacing $G_{r,n+1}(t,q)$, hence the simple real-rootedness of $G_{r,n}(t,q)$ for all $n\geqslant 1$.

Conjecture 13. *Let $r,n\geqslant 1$ and $q>0$. Suppose that $G_{r,n}(t,q)$ is simply real-rooted and let $t^r_{n,1}(q)<t^r_{n,2}(q)<\cdots<t^r_{n,n}(q)<0$ be these real zeros. Then $t^r_{n,i}(q)$ satisfy the following separation property*:

$$t^r_{n,i+1}(q)>\min\big(q^r,q^{-r}\big)t^r_{n,i}(q),\quad i=1,2,\ldots,n-1.$$

6. Concluding remarks

In case $r=1,2$, $G_{r,n}$ are respectively the symmetric group $\mathfrak{S}_n$ and the hyperoctahedral group B_n, which are finite Coxeter groups. It is well known that for finite Coxeter groups, statistics equidistributed with the length function are termed Mahonian. For $r>2$, $G_{r,n}$ is no longer a Coxeter group. However, the coincidence of distribution of a statistic with the Poincaré polynomial serves as the defining condition for the statistic to be Mahonian.

Analogous to types A and B cases, we define a sequence of polynomials $\{P_{r,n}(t,q)\}$ in two variables to be r-Euler–Mahonian if $P_{r,n}(t,q)$ satisfies any part of Theorem 9. We further define that a pair of statistics $(\mathrm{stat}_1,\mathrm{stat}_2)$ on the wreath product $G_{r,n}$ to be r-Euler–Mahonian if

$$\sum_{p\in G_{r,n}} t^{\mathrm{stat}_1(p)}q^{\mathrm{stat}_2(p)}=G_{r,n}(t,q).$$

It is clear that 1- and 2-Euler–Mahonian pairs are exactly the types A and B Euler–Mahonian pairs, respectively. Bagno and Biagioli [5] recently consider the descent representations of $G_{r,p,n}$ and obtain, by a specialization of the multigraded Hilbert series of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$ of complex polynomials in $x_1,x_2,\ldots,x_n$ by exponent partitions, a generalized Carlitz identity for $G_{r,n}$, namely,

$$\frac{\sum_{g\in G_{r,n}} t^{\mathrm{fdes}(g)}q^{\mathrm{fmaj}(g)}}{(1-t)(1-t^rq^r)(1-t^rq^{2r})\cdots(1-t^rq^{nr})}=\sum_{k\geqslant 0}[k+1]_q^n t^k,$$

where $\mathrm{fdes}(g)$ and $\mathrm{fmaj}(g)$ are the flag descent number and flag major index of $g \in G_{r,n}$. The latter flag major index is different from the one defined in Section 3 here. In light of the present work, it would be interesting to find the multigraded Hilbert series corresponding to our choice of $(\mathrm{des}_r, \mathrm{fmaj}_r)$ and to realize Theorem 9(iv) as a specialization of it. This will be the subject of further research.

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